Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 227 (2011) 1772–1805

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

The quaternionic evolution operator

Fabrizio Colombo *, Irene Sabadini

Politecnico di Milano, Dipartimento di Matematica, Via E. Bonardi, 9, 20133 Milano, Italy

Received 9 March 2011; accepted 6 April 2011

Available online 20 April 2011

Communicated by Edoardo Vesentini

Abstract

In the recent years, the notion of slice regular functions has allowed the introduction of a quaternionic functional calculus. In this paper, motivated also by the applications in quaternionic quantum mechanics, see Adler (1995) [1], we study the quaternionic semigroups and groups generated by a quaternionic (bounded or unbounded) linear operator $T = T_0 + iT_1 + jT_2 + kT_3$. It is crucial to note that we consider operators with components T_ℓ ($\ell = 0, 1, 2, 3$) that do not necessarily commute. Among other results, we prove the quaternionic version of the classical Hille–Phillips–Yosida theorem. This result is based on the fact that the Laplace transform of the quaternionic semigroup e^{tT} is the S-resolvent operator $(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(\bar{s}\mathcal{I} - T)$, the quaternionic analogue of the classical resolvent operator. The noncommutative setting entails that the results we obtain are somewhat different from their analogues in the complex setting. In particular, we have four possible formulations according to the use of left or right slice regular functions for left or right linear operators.

MSC: 47A10; 47A60

Keywords: Right and left linear quaternionic operators; Quaternionic semigroup; Quaternionic group; Bounded and unbounded quaternionic generators; Hille–Phillips–Yosida theorem in the quaternionic setting; S-resolvent operator; S-spectrum

* Corresponding author.
E-mail addresses: fabrizio.colombo@polimi.it (F. Colombo), irene.sabadini@polimi.it (I. Sabadini).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.04.001

1. Introduction

In the past few years it has been possible to introduce a quaternionic functional calculus and a functional calculus for n-tuples of noncommuting operators based on the theory of slice hyperholomorphic functions. By this term we mean either slice regular functions of a quaternionic variable, see [19], or slice monogenic functions of a paravector variable with values in a Clifford algebra, see [14]. It is interesting to note that, on one hand, the theory of slice hyperholomorphic functions allowed the definition of these functional calculi, on the other hand, the introduction of these calculi gave an impulse to further develop the theory of these functions, see [5,8,9,11,16].

In this paper we focus our attention on slice regular functions. It is crucial that these functions present striking differences with respect to the classical Fueter regular functions. For example, polynomials and series in the quaternionic variable q are slice regular, but they are not regular in the sense of Fueter. Moreover, the Cauchy formulas for the left and for the right slice regular functions involve different Cauchy kernels, unlike what happens in the classical case. Despite the fact that Fueter regular functions have had a great success in the past decades, as Adler has noticed at the end of Chapter 1 of his book [1], they are not a natural tool when dealing with quaternionic quantum mechanics. Slice regularity seems to be the right notion of analyticity to be used in that framework, indeed it allows to define functions, like the exponential, of quaternionic operators. Moreover, the quaternionic functional calculus based on slice regularity allows to deal with quaternionic linear operators whose components do not necessarily commute. Note that the calculus has been introduced in [6,7] in a special case, i.e. using the subclass of slice regular functions admitting power series expansions with center at real points. Later on, we have extended the validity of the functional calculus to the general case, and we have studied its main properties, see [9]. Furthermore, the quaternionic functional calculus has been further developed in four different cases: for left and right linear quaternionic operators, and using left or right slice regular functions. These four cases present analogies but also some differences which are thoroughly discussed in [10].

The quaternionic evolution operator (for right bounded linear operators) has been originally introduced in [9], where we provided its relation with the Laplace transform. As it is well known, this transform is an important tool to deal with semigroup theory, a wide field of study that we consider from the point of view of the approach developed by E. Hille, R.S. Phillips and K. Yosida. Their results, in particular, imply that if a semigroup $E(t)$ is continuous in the uniform operator topology, then there exists a bounded linear operator B, acting on a Banach space, such that $E(t) = e^{tB}$, see e.g. [18] and [25]. Conversely, from the Riesz–Dunford functional calculus, it easily follows that, for any bounded linear operator B, the operator e^{tB} is a uniformly continuous semigroup. Moreover, the generator B can be obtained as the limit $\lim_{h\to 0}(E(h) - I)/h$ and the Laplace transform of the semigroup, for $\mathrm{Re}[\lambda]$ suitable large, gives the resolvent operator, that is:

$$(\lambda I - B)^{-1} = \int_0^\infty e^{-\lambda t} E(t)\, dt.$$

If we consider $E(t)$ which is merely continuous in the strong operator topology, then the problem becomes more difficult to study. In an appropriate sense e^{tB} still equals $E(t)$, but now B is an unbounded linear operator called the infinitesimal generator of the semigroup. In fact, for closed densely defined operators, the characterization is given by:

Theorem 1.1. *A necessary and sufficient condition in order that a closed linear operator B with dense domain be the infinitesimal generator of a strongly continuous semigroup is that there exist real numbers $M > 0$ and ω such that for every real number $\lambda > \omega$, λ is in the resolvent set of B and $\|(\lambda I - B)^{-n}\| \leqslant M(\lambda - \omega)^{-n}$, for $n \in \mathbb{N}$.*

We will show how these results can be generalized to the quaternionic setting. Our theory allows also to define a notion of Laplace transform which gives the right or the left S-resolvent operators according to the fact that we are using the functional calculus for left or for right slice regular functions. Indeed, if T is a bounded quaternionic linear operator on a bilateral quaternionic Banach space V then, for $\mathrm{Re}[s] > \|T\|$, the left S-resolvent operator ($S_L^{-1}(s,T) := -(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I})$) is given by

$$-\big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)^{-1}(T - \bar{s}\mathcal{I}) = \int_0^{+\infty} e^{tT} e^{-ts}\, dt,$$

while the right S-resolvent operator ($S_R^{-1}(s,T) := -(T - \bar{s}\mathcal{I})(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}$), is given by

$$-(T - \bar{s}\mathcal{I})\big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)^{-1} = \int_0^{+\infty} e^{-ts} e^{tT}\, dt.$$

Finally, we recall that the use of noncommutative structures such as Clifford algebras gives rise to a functional calculus for n-tuples of noncommuting operators. In this setting we have two possible approaches. The first one is based on the classical monogenic functions, see [3,12], and is related with the results illustrated in the book of B. Jefferies [21] and, for example, in the papers [22–24]. The second one, more recent, is based on slice monogenic functions (see [8,11,14,15] and their generalization given in [20]) and it is introduced and studied in the papers [11,13] and in the forthcoming book [17].

We conclude by describing the structure of the paper in which we will mainly treat the case of right linear operators, which is the most studied in the literature. The preliminary material on slice regularity and on the quaternionic functional calculus is described in Section 2. The main results of Section 3 treat the case of uniformly continuous quaternionic semigroups, while in Section 4 we investigate the case of strongly continuous quaternionic semigroups. In Section 5 we discuss the case of quaternionic groups. In Section 6 we discuss the use of the functional calculus associated to right regular functions in the definition of the evolution operator and some consequences. In Section 7 we also mention some of the modifications of our theory to treat the evolution operator in the case of a left linear quaternionic operator.

2. Preliminary results

The class of functions for which it is possible to define our quaternionic functional calculus, based on the general Cauchy formula proved in [5], are the slice regular functions introduced in [19] and further studied, for example, in [4]. The Cauchy formula is a crucial tool for the theory: it has been proved for left slice monogenic functions in [8], while the version for right monogenic functions is in [10].

Let us start by recalling the basic definitions. The canonical basis for the real associative algebra of quaternions $\mathbb{H}$ is given by the elements $1, i, j, k$ satisfying the relations

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

A quaternion q is denoted by $q = x_0 + ix_1 + jx_2 + kx_3$, $x_\ell \in \mathbb{R}$, $\ell = 0, 1, 2, 3$, its conjugate is $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$, while its norm is given by $|q|^2 = q\bar{q}$. The real part of a quaternion will be denoted either by the symbol $\mathrm{Re}[q]$ or by x_0. Let $\mathbb{S}$ be the 2-dimensional sphere of purely imaginary unit quaternions, i.e.

$$\mathbb{S} = \{ q = ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \}.$$

To each quaternion p it is possible to associate an element on the sphere $\mathbb{S}$:

$$I_p = \begin{cases} \frac{\mathrm{Im}[p]}{|\mathrm{Im}[p]|} & \text{if } \mathrm{Im}[p] \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}$$

Given $I \in \mathbb{S}$ we denote by $\mathbb{C}_I$ the complex plane $\mathbb{R} + I\mathbb{R}$ containing elements of the form $x + Iy$, $x, y \in \mathbb{R}$. Obviously, the imaginary unit I_p determines the complex plane $\mathbb{C}_{I_p}$ containing p.

Definition 2.1. Given $p \in \mathbb{H}$, $p = p_0 + I_p p_1$ we denote by $[p]$ the set of all elements of the form $p_0 + Jp_1$ when J varies in $\mathbb{S}$.

The set $[p]$ is a 2-sphere which is reduced to the point p when $p \in \mathbb{R}$.

Definition 2.2 *(Slice regular functions).* (See [19].) Let $U \subseteq \mathbb{H}$ be an open set and let $f : U \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$.

- We say that f is a *left slice regular* (for short, *regular*) function if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

  We will denote by $\mathcal{R}^L(U)$ the right $\mathbb{H}$-vector space of left regular functions on the open set U.
- We say that f is *right slice regular* (for short, *right regular*) function if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy) I\right) = 0.$$

  We will denote by $\mathcal{R}^R(U)$ the left $\mathbb{H}$-vector space of right regular functions on the open set U.

In order to state the Cauchy formula, we have to identify the domains on which regular functions are naturally defined. As we shall see, these domains are also involved in the definition of the quaternionic functional calculus.

Definition 2.3. Let U be a domain in $\mathbb{H}$. We say that U is a *slice domain* (s-domain for short) if $U \cap \mathbb{R}$ is nonempty and if $U \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$. We say that U is *axially symmetric* if, for all $q = x + Iy \in U$, the 2-sphere $x + Jy$, $J \in \mathbb{S}$, is contained in U.

Theorem 2.4 *(The left and right Cauchy formulas). (See [5,16].) Let W be an open set in $\mathbb{H}$. Let $\overline{U} \subset W$ be an axially symmetric s-domain, and let $\partial(U \cap \mathbb{C}_I)$ be the union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Set $ds_I = ds/I$.*

- *Let $f \in \mathcal{R}^L(W)$. Then, if $q \in U$, we have*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s,q)\, ds_I\, f(s) \tag{1}$$

  and the integral (1) *does not depend on the choice of the imaginary unit $I \in \mathbb{S}$ nor on U. The so-called left kernel*

$$S_L^{-1}(s,q) = -\left(q^2 - 2q\,\mathrm{Re}[s] + |s|^2\right)^{-1}(q - \bar{s})$$

  is a left regular function in the variable q and it is right regular with respect to the variable s.
- *Let $f \in \mathcal{R}^R(W)$. Then, if $q \in U$, we have*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s,q) \tag{2}$$

  and the integral (2) *does not depend on the choice of the imaginary unit $I \in \mathbb{S}$ nor on U. The so-called right kernel*

$$S_R^{-1}(s,q) := -(q - \bar{s})\left(q^2 - 2\,\mathrm{Re}[s]q + |s|^2\right)^{-1}$$

  is a right regular function in the variable q and it is left regular with respect to the variable s.

2.1. The functional calculus for bounded linear quaternionic operators

Let V be a bilateral vector space on $\mathbb{H}$. A map $T : V \to V$ is said to be a right linear operator if

$$T(u + v) = T(u) + T(v), \qquad T(us) = T(u)s, \quad \text{for all } s \in \mathbb{H}, \text{ and for all } u, v \in V.$$

The set of right linear operators on V is both a left and a right vector space on $\mathbb{H}$ with respect to the operations

$$(sT)(v) := sT(v), \qquad (Ts)(v) := T(sv), \quad \text{for all } s \in \mathbb{H}, \text{ and for all } v \in V.$$

The quaternionic identity operator will be denoted by $\mathcal{I}$.

Remark 2.5. In the sequel we will always consider bilateral quaternionic Banach spaces even when for sake of brevity we sometimes omit the word "bilateral". Moreover, we will limit ourselves to the case of right linear operators because it is the most used case in literature. It is obviously possible to define the quaternionic evolution operators for left linear operators, with suitable modifications of the theory. We discuss this possibility in Section 7.

Let V be a Banach space over $\mathbb{H}$. We will denote by $\mathcal{B}(V)$ the bilateral vector space of all right linear bounded quaternionic operators on V. It is easy to verify that $\mathcal{B}(V)$ is a Banach space endowed with its natural norm.

Let us introduce the main concepts to define the quaternionic functional calculus.

Definition 2.6 *(The S-spectrum and the S-resolvent sets for quaternionic operators).* Let $T \in \mathcal{B}(V)$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T) = \big\{s \in \mathbb{H}\colon\ T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I} \text{ is not invertible in } \mathcal{B}(V)\big\},$$

where $\mathrm{Re}[s]$, $|s|$ denote the real part and the modulus of the quaternion s, respectively. The S-resolvent set $\rho_S(T)$ is defined by

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

Definition 2.7 *(The S-resolvent operator).* Let $T \in \mathcal{B}(V)$. We define the left S-resolvent operator as

$$S_L^{-1}(s,T) := -\big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)^{-1}(T - \bar{s}\mathcal{I}), \tag{3}$$

and the right S-resolvent operator as

$$S_R^{-1}(s,T) := -(T - \bar{s}\mathcal{I})\big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)^{-1}. \tag{4}$$

The following properties are crucial in the definition of the quaternionic functional calculus (see [6,9,10]).

Theorem 2.8 *(The S-resolvent equation). Let $T \in \mathcal{B}(V)$ and $s \in \rho_S(T)$, then the left S-resolvent operator, defined in* (3)*, satisfies the equation*

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I}, \tag{5}$$

while the right S-resolvent operator, defined in (4)*, satisfies the equation*

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}. \tag{6}$$

We are now in the position to introduce the admissible functions to define the quaternionic functional calculus.

Definition 2.9. Let $T \in \mathcal{B}(V)$ and let $U \subset \mathbb{H}$ be an axially symmetric s-domain that contains the S-spectrum $\sigma_S(T)$. Suppose that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$.

- A function $f \in \mathcal{R}^L(W)$ (resp. $f \in \mathcal{R}^R(W)$), where W is an open set in $\mathbb{H}$, is said to be locally regular on $\sigma_S(T)$ if there exists a domain $U \subset \mathbb{H}$ as above such that $\overline{U} \subset W$.
- We will denote by $\mathcal{R}^L_{\sigma_S(T)}$ the set of locally left regular functions on $\sigma_S(T)$ and by $\mathcal{R}^R_{\sigma_S(T)}$ the set of locally right regular functions on $\sigma_S(T)$.

Remark 2.10. Provided that the elements in the S-spectrum are 2-spheres (see [6]), the fact to consider axially symmetric s-domains is not reductive.

Definition 2.11. Let V be a bilateral quaternionic Banach space, $T \in \mathcal{B}(V)$, $U \subset \mathbb{H}$ be a domain as in Definition 2.9. Choose $I \in \mathbb{S}$ and set $ds_I = ds/I$. If $f \in \mathcal{R}^L_{\sigma_S(T)}$ we define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, T)\, ds_I\, f(s). \tag{7}$$

If $f \in \mathcal{R}^R_{\sigma_S(T)}$ we define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s, T). \tag{8}$$

Remark 2.12. The quaternionic functional calculus is well defined since the integrals in (7) and (8) do not depend on U nor on $I \in \mathbb{S}$.

2.2. *The S-resolvent operators for closed linear quaternionic operators*

As we pointed out in the introduction, there are two formulations, left and right, of the quaternionic functional calculus. In this subsection we discuss the main differences between them which will be important to study the evolution operator. All the missing proofs of the results in this subsection are in [10].

Definition 2.13. Let V be a quaternionic Banach space. We denote by $\mathcal{K}(V)$ the set of right linear closed operators $T : \mathcal{D}(T) \subset V \to V$, such that

(1) $\mathcal{D}(T)$ is dense in V,
(2) $\mathcal{D}(T^2) \subset \mathcal{D}(T)$ is dense in V,
(3) $T - \bar{s}\mathcal{I}$ is densely defined in V.

Since T is a closed operator then $T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I} : \mathcal{D}(T^2) \subset V \to V$ is a closed operator. If $T \in \mathcal{K}(V)$ we denote by $\rho_S(T)$ the S-resolvent set of T as

$$\rho_S(T) = \left\{ s \in \mathbb{H}:\ \left(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\right)^{-1} \in \mathcal{B}(V) \right\}.$$

We define the S-spectrum $\sigma_S(T)$ of T as $\sigma_S(T) = \mathbb{H} \setminus \rho_S(T)$.

In the sequel we will use the following notation:

Definition 2.14. Let $T \in \mathcal{K}(V)$ and $s \in \rho_S(T)$. We denote by $Q_s(T)$ the operator:

$$Q_s(T) := \big(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\big)^{-1} \quad \text{where } Q_s(T): V \to \mathcal{D}\big(T^2\big). \tag{9}$$

The definition of S-resolvent operator in the unbounded case relies on a difference between the left and right formulations of the quaternionic functional calculus. Recall that, for $s \in \rho_S(T)$, the left S-resolvent operator in the bounded case is

$$S_L^{-1}(s,T) = -Q_s(T)(T - \bar{s}\mathcal{I}). \tag{10}$$

Now observe that in the case T is unbounded $S_L^{-1}(s,T)$ turns out to be defined only on $\mathcal{D}(T)$. This fact is due to the term $Q_s(T)T$. However, the operator $Q_s(T)T$ is the restriction to the dense subspace $\mathcal{D}(T)$ of V of a bounded linear operator defined on V. This fact follows by the commutation relation $Q_s(T)Tv = TQ_s(T)v$ which holds for all $v \in \mathcal{D}(T)$ since the polynomial operator $T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}: \mathcal{D}(T^2) \to V$ has real coefficients. The operator $TQ_s(T)$ acts on V, more precisely, $TQ_s(T): V \to \mathcal{D}(T)$ and it is continuous for $s \in \rho_S(T)$. This fact motivates the following definition.

Definition 2.15. Let A be an operator containing the term $Q_s(T)T$ (resp. $TQ_s(T)$) and let $s \in \rho_S(T)$. We define $\hat{A}$ to be the operator obtained from A by substituting each occurrence of $Q_s(T)T$ (resp. $TQ_s(T)$) by $TQ_s(T)$ (resp. $Q_s(T)T$).

Definition 2.16 *(The S-resolvent operators for the unbounded case).* Let $T \in \mathcal{K}(V)$ and $s \in \rho_S(T)$. We define the left S-resolvent operator as

$$S_L^{-1}(s,T)v := -Q_s(T)(T - \bar{s}\mathcal{I})v, \quad v \in \mathcal{D}(T), \tag{11}$$

and we will call

$$\widehat{S_L^{-1}}(s,T)v = Q_s(T)\bar{s}v - TQ_s(T)v, \quad v \in V, \tag{12}$$

the extended left S-resolvent operator. We define the right S-resolvent operator as

$$S_R^{-1}(s,T)v := -(T - \bar{s}\mathcal{I})Q_s(T)v, \quad v \in V. \tag{13}$$

Remark 2.17. For the right S-resolvent operator $S_R^{-1}(s,T)$ we have that the operator $Q_s(T)$: $V \to \mathcal{D}(T^2)$ is bounded for $s \in \rho_S(T)$, so also $(T - \bar{s}\mathcal{I})Q_s(T): V \to \mathcal{D}(T)$ is bounded.

The second difference between the left and the right functional calculus are the S-resolvent equations which, to hold on V, need different extensions of the operators involved, see [10].

Theorem 2.18 *(The S-resolvent equations). Let $T \in \mathcal{K}(V)$ and $s \in \rho_S(T)$. The left S-resolvent operator, defined in* (11)*, satisfies the equation*

$$S_L^{-1}(s,T)sv - TS_L^{-1}(s,T)v = \mathcal{I}v, \quad v \in \mathcal{D}(T), \tag{14}$$

moreover the extended left S-resolvent operator (12) *satisfies*

$$\widehat{S}_L^{-1}(s,T)sv - T\widehat{S}_L^{-1}(s,T)v = \mathcal{I}v, \quad v \in V. \tag{15}$$

The right S-resolvent operator, defined in (13)*, satisfies the equation*

$$sS_R^{-1}(s,T)v - S_R^{-1}(s,T)Tv = \mathcal{I}v, \quad v \in \mathcal{D}(T), \tag{16}$$

moreover

$$sS_R^{-1}(s,T)v - \big(S_R^{-1}\widehat{(s,T)T}\big)v = \mathcal{I}v, \quad v \in V. \tag{17}$$

3. Uniformly continuous quaternionic semigroups

In this section we generalize to the quaternionic setting the classical result saying that a semigroup has a bounded infinitesimal generator if and only if it is uniformly continuous. To start with, we recall the definition of uniformly continuous and of strongly continuous semigroups and some preliminary results which will be useful in the sequel. Note that to develop our theory we will make use of the functional calculus based on left regular functions. The corresponding results for right regular functions will be treated in Section 3.1.

Definition 3.1. Let V be a bilateral quaternionic Banach space and $t \in \mathbb{R}$. A family $\{\mathcal{U}(t)\}_{t\geqslant 0}$ of linear bounded quaternionic operators in V will be called a *strongly continuous quaternionic semigroup* if

(1) $\mathcal{U}(t+\tau) = \mathcal{U}(t)\mathcal{U}(\tau)$, $t, \tau \geqslant 0$,
(2) $\mathcal{U}(0) = \mathcal{I}$,
(3) for every $v \in V$, $\mathcal{U}(t)v$ is continuous in $t \in [0, \infty]$.

If, in addition,

(4) the map $t \to \mathcal{U}(t)$ is continuous in the uniform operator topology,

then the family $\{\mathcal{U}(t)\}_{t\geqslant 0}$ is called a *uniformly continuous quaternionic semigroup* in $\mathcal{B}(V)$.

From the functional calculus in Definition 2.11, it is clear that for any $T \in \mathcal{B}(V)$, the operator e^{tT} is a uniformly continuous quaternionic semigroup in $\mathcal{B}(V)$. The following theorem shows that also the converse is true, i.e. every uniformly continuous quaternionic semigroup is of this form.

Theorem 3.2. *Let $\{\mathcal{U}(t)\}_{t\geqslant 0}$ be a uniformly continuous quaternionic semigroup in $\mathcal{B}(V)$. Then:*

(1) *there exists a bounded linear quaternionic operator T such that $\mathcal{U}(t) = e^{tT}$;*
(2) *the quaternionic operator T is given by the formula*

$$T = \lim_{h\to 0} \frac{\mathcal{U}(h) - \mathcal{U}(0)}{h};$$

(3) *we have the relation:*

$$\frac{d}{dt}e^{tT} = Te^{tT} = e^{tT}T.$$

Proof. We start by proving point (1). Let us consider the logarithmic function $\ln q$ defined on $\mathbb{H} \setminus \{q \in \mathbb{R}\colon q \leqslant 0\}$ by extending the principal branch of the function $\ln q$. Since $\mathcal{U}(0) = \mathcal{I}$ whose S-spectrum is reduced to the real point 1, it follows that we can apply the perturbation theorem of the S-resolvent operator, see Theorem 4.7 in [10], to the operators $\mathcal{U}(0) = \mathcal{I}$ and $\mathcal{U}(\delta)$ for a suitable $\delta > 0$, using the function $\ln q$. Thus, there exists $\varepsilon > 0$ such that $P(t) = \ln\mathcal{U}(t)$ is defined and continuous for $t \in [0, \varepsilon]$. If $nt \leqslant \varepsilon$ then, for the semigroup properties, we have

$$P(nt) = \ln\mathcal{U}(nt) = \ln\bigl(\mathcal{U}(t)\bigr)^n = nP(t),$$

thus

$$P(t) = nP(t/n) \quad \text{for every } t \in [0, \varepsilon].$$

As a consequence, for each rational number m/n such that $m/n \in [0, 1]$ and for each $t \in [0, \varepsilon]$, we have

$$\frac{m}{n}\mathcal{U}(t) = m\mathcal{U}(t/n) = \mathcal{U}(mt/n),$$

and so

$$\frac{m}{n}\mathcal{U}(\varepsilon) = \mathcal{U}(m\varepsilon/n).$$

By continuity, we get

$$tP(\varepsilon) = P(t\varepsilon) \quad \text{for every } t \in [0, 1],$$

and

$$P(t) = \frac{t}{\varepsilon}P(\varepsilon) \quad \text{for every } t \in [0, \varepsilon].$$

If we set

$$T := \frac{1}{\varepsilon}P(\varepsilon),$$

we obtain

$$\mathcal{U}(t) = e^{tT} \quad \text{for every } t \in [0, \varepsilon].$$

If $t > 0$ is arbitrary then $t/n < \varepsilon$ for sufficiently large n, and so we obtain

$$e^{tT} = \bigl(e^{(t/n)T}\bigr)^n = \bigl[\mathcal{U}(t/n)\bigr]^n = \mathcal{U}(t).$$

This proves the representation of the semigroup. To prove point (2) let $h > 0$ and observe that the limit $\lim_{h\to 0^+}(e^{hq} - 1)/h = q$ and the sequence $(e^{hq} - \mathcal{I})/h$ converges uniformly in any bounded set of $\mathbb{H}$. So, by Theorem 4.3 in [9], $\lim_{h\to 0^+}(e^{hT} - \mathcal{I})/h$ converges to T. Point (3) can be deduced by the functional calculus. $\square$

We now want to generalize the important result that the Laplace transform of a semigroup e^{tB} of bounded linear complex operator B is the resolvent operator $(\lambda I - B)^{-1}$. The generalization we obtain is somewhat surprising. Both the left and the right S-resolvent operators $S_L^{-1}(s,T)$ and $S_R^{-1}(s,T)$ are the Laplace transform of the semigroup according to the two different possible definitions of the Laplace transform based on the left or on the right regular functions. In the case of $S_L^{-1}(s,T)$ we have (see [9]):

Theorem 3.3. *Let $T \in \mathcal{B}(V)$ and let $s_0 > \|T\|$. Then the left S-resolvent operator $S_L^{-1}(s,T)$ is given by*

$$S_L^{-1}(s,T) = \int_0^{+\infty} e^{tT} e^{-ts}\, dt. \tag{18}$$

The case of $S_R^{-1}(s,T)$ will be treated in Theorem 3.12 at the end of this section.

We now state and prove a result which will be useful in the sequel:

Proposition 3.4. *Let V be a quaternionic Banach space. Let $\{\mathcal{U}(t)\}$ be a family of bounded linear quaternionic operators defined on a finite closed interval $[a,b]$ such that $\mathcal{U}(t)v$ is continuous in t for each $v \in V$; then $\|\mathcal{U}(\cdot)\|$ is measurable and bounded on $[a,b]$. Conversely, if $\{\mathcal{U}(t)\}_{t\geqslant 0}$ is a semigroup of bounded linear quaternionic operators in V and if $\mathcal{U}(\cdot)v$ is measurable on $(0,\infty)$ for each $v \in V$, then $\mathcal{U}(\cdot)v$ is continuous at every point in $(0,\infty)$.*

Proof. The statement follows by adapting the arguments in the proof of Lemma 3, p. 616 in [18]. In fact, under the hypotheses in the first part of the statement, the boundedness of $\|\mathcal{U}(\cdot)\|$ follows from the Uniform Boundedness Principle and the fact that $\|\mathcal{U}(\cdot)\|$ is measurable follows from Theorem III.6.10 in [18]. To show the second part, we can assume that $\|\mathcal{U}(\cdot)\|$ is bounded over each interval of the form $[\delta, 1/\delta]$, $\delta > 0$. Under this assumption, if one repeats the computations in the proof of Lemma 3, p. 616 in [18], one gets that $\|\mathcal{U}(\cdot)\|$ is continuous at each point $t_0 > 0$ for any $v \in V$. Finally, it is sufficient to show that, if $\|\mathcal{U}(\cdot)\|v$ is measurable on $(0,\infty)$ for all $v \in V$, then it is bounded on $[\delta, 1/\delta]$, $\delta > 0$. $\square$

We now recall two well-known results, see [18], which depend only on the norm of an operator and thus they hold also in the quaternionic setting.

Proposition 3.5. *Let $\{\mathcal{U}(t)\}_{t\geqslant 0}$ be a family of bounded linear quaternionic operators on the quaternionic Banach space V. If*

$$p(t) := \ln\|\mathcal{U}(t)\|$$

is bounded from the above on the interval $(0, a)$ *for every positive* $a \in \mathbb{R}$*, then*

$$\lim_{t \to +\infty} t^{-1} \ln \|\mathcal{U}(t)\| = \inf_{t>0} t^{-1} \ln \|\mathcal{U}(t)\|.$$

A direct consequence of Proposition 3.5 is the following important result.

Proposition 3.6. *Let* $\{\mathcal{U}(t)\}_{t \geqslant 0}$ *be a family of bounded linear quaternionic operators on a quaternionic Banach space* V*. Then:*

(1) *the limit* $\omega_0 := \lim_{t \to +\infty} t^{-1} \ln \|\mathcal{U}(t)\|$ *exists;*
(2) *for each* $\delta > \omega_0$ *there exists a positive constant* M_δ *such that* $\|\mathcal{U}(t)\| \leqslant M_\delta e^{\delta t}$, $\forall t \geqslant 0$.

We are now ready to give the following definition.

Definition 3.7 *(Quaternionic infinitesimal generator).* Let $\{\mathcal{U}(t)\}_{t \geqslant 0}$ be a family of bounded linear quaternionic operators on a quaternionic Banach space V.

(1) For each $h > 0$ define the linear quaternionic operator

$$T_h v = \frac{\mathcal{U}(h)v - v}{h}, \quad v \in V.$$

(2) Set $\mathcal{D}(T) := \{v \in V\colon \lim_{h \to 0^+} T_h v \text{ exists in } V\}$ and define the quaternionic operator T with domain $\mathcal{D}(T)$ by the formula

$$Tv = \lim_{h \to 0^+} T_h v, \quad v \in \mathcal{D}(T).$$

The operator T, with domain $\mathcal{D}(T)$, is called the infinitesimal quaternionic generator of the quaternionic semigroup $\mathcal{U}(t)$.

We have the following properties:

Proposition 3.8. *Let* T *be the infinitesimal quaternionic generator of the quaternionic semigroup* $\mathcal{U}(t)$ *and let* $\mathcal{D}(T)$ *be its domain. Then:*

(1) *the set* $\mathcal{D}(T)$ *is a linear subspace of* V *and* T *is linear on* $\mathcal{D}(T)$*;*
(2) *if* $v \in V$ *then* $\mathcal{U}(t)v \in \mathcal{D}(T)$ *for* $t \geqslant 0$*. Moreover,*

$$\frac{d}{dt}\mathcal{U}(t)v = T\mathcal{U}(t)v = \mathcal{U}(t)Tv, \quad v \in \mathcal{D}(T);$$

(3) *if* $v \in \mathcal{D}(T)$*, then*

$$\mathcal{U}(t)v - \mathcal{U}(\tau)v = \int_\tau^t \mathcal{U}(\theta)Tv\, d\theta, \quad 0 \leqslant \tau < t < \infty;$$

(4) *let* $g:[0,\infty]\to\mathbb{H}$ *be a Lebesgue integrable function, continuous at* $t\in[0,\infty]$*, then*

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}\mathcal{U}(\theta)g(\theta)v\,d\theta=\mathcal{U}(t)g(t)v.$$

Proof. Point (1) follows from the definition. Let us show point (2). Set $h>0$, $t\geqslant 0$ and $v\in\mathcal{D}(T)$. Then we can write

$$\mathcal{U}(t)T_h v=T_h\mathcal{U}(t)v,$$

passing to the limit

$$\lim_{h\to 0^+}\mathcal{U}(t)T_h v=\lim_{h\to 0^+}T_h\mathcal{U}(t)v$$

we have that $\mathcal{U}(t)v\in\mathcal{D}(T)$, so by definition

$$T\mathcal{U}(t)v=\lim_{h\to 0^+}T_h\mathcal{U}(t)v,$$

thus

$$\mathcal{U}(t)Tv=T\mathcal{U}(t)v,\quad v\in\mathcal{D}(T).$$

This proves that $\mathcal{U}(t)v\in\mathcal{D}(T)$, for all $t\geqslant 0$. If $t>0$ and $h>0$ then, consider the limit

$$L=\lim_{h\to 0^+}\left(\frac{\mathcal{U}(t)v-\mathcal{U}(t-h)v}{h}-\mathcal{U}(t)Tv\right),$$

by the semigroup properties and the definition of T_h we have

$$\begin{aligned}
&\frac{\mathcal{U}(t)v-\mathcal{U}(t-h)v}{h}-\mathcal{U}(t)Tv\\
&\quad=\mathcal{U}(t-h)\frac{\mathcal{U}(h)v-v}{h}-\mathcal{U}(t)Tv\\
&\quad=\mathcal{U}(t-h)\frac{\mathcal{U}(h)v-v}{h}-\mathcal{U}(t-h)Tv+\mathcal{U}(t-h)Tv-\mathcal{U}(t)Tv\\
&\quad=\mathcal{U}(t-h)(T_hv-Tv)+\big[\mathcal{U}(t-h)-\mathcal{U}(t)\big]Tv.
\end{aligned}$$

Consider the limit for $h\to 0^+$: since the semigroup is uniformly continuous in $\mathcal{B}(V)$ by Proposition 3.4, we obtain $L=0$. On the other hand, we have that

$$\frac{\mathcal{U}(t+h)v-\mathcal{U}(t)v}{h}=\mathcal{U}(t)T_hv,$$

taking the limit for $h\to 0^+$ we obtain

$$\frac{d}{dt}\mathcal{U}(t)v=T\mathcal{U}(t)v=\mathcal{U}(t)Tv,\quad \text{for all } v\in\mathcal{D}(T).$$

We have thus proved that the derivation formula holds for all $v \in \mathcal{D}(T)$. To prove point (3), observe that point (2) implies that

$$\left\langle \varphi, \frac{d}{d\tau}\mathcal{U}(\tau)v \right\rangle = \langle \varphi, \mathcal{U}(\tau)Tv \rangle$$

for all linear and continuous functionals $\varphi \in V'$. By integrating

$$\int_s^t \left\langle \varphi, \frac{d}{d\tau}\mathcal{U}(\tau)v \right\rangle d\tau = \int_s^t \langle \varphi, \mathcal{U}(\tau)Tv \rangle \, d\tau$$

we have

$$\langle \varphi, \mathcal{U}(t)v - \mathcal{U}(s)v \rangle = \left\langle \varphi, \int_s^t \mathcal{U}(\tau)Tv \, d\tau \right\rangle \quad \text{for all } \varphi \in V',$$

from which we deduce (3). Finally, point (4) follows from Theorem III.12.8 in [18] which holds also in this setting, with obvious modifications. □

Lemma 3.9. *The linear subspace $\mathcal{D}(T)$ as in Definition* 3.7 *is dense in V and T is closed on $\mathcal{D}(T)$.*

Proof. Let T_h be as in Definition 3.7, take $v \in V$ and for $h > 0$ and $t > 0$ consider

$$\begin{aligned}
T_h \int_0^t \mathcal{U}(\tau)v \, d\tau &= \frac{1}{h} \int_0^t \big[\mathcal{U}(h+\tau)v - \mathcal{U}(\tau)v\big] \, d\tau \\
&= \frac{1}{h} \int_h^{h+t} \mathcal{U}(\tau)v \, d\tau - \frac{1}{h} \int_0^t \mathcal{U}(\tau)v \, d\tau \\
&= \frac{1}{h} \int_t^h \mathcal{U}(\tau)v \, d\tau + \frac{1}{h} \int_h^{h+t} \mathcal{U}(\tau)v \, d\tau - \frac{1}{h} \int_0^t \mathcal{U}(\tau)v \, d\tau - \frac{1}{h} \int_t^h \mathcal{U}(\tau)v \, d\tau \\
&= \frac{1}{h} \int_t^{h+t} \mathcal{U}(\tau)v \, d\tau - \frac{1}{h} \int_0^h \mathcal{U}(\tau)v \, d\tau.
\end{aligned}$$

By Proposition 3.8 point (4) we get

$$\lim_{h \to 0^+} T_h \int_0^t \mathcal{U}(\tau)v \, d\tau = \mathcal{U}(t)v - v,$$

so $\int_0^t \mathcal{U}(\tau)v\,d\tau \in \mathcal{D}(T)$ and since

$$v = \lim_{t\to 0^+} \frac{1}{t}\int_0^t \mathcal{U}(\tau)v\,d\tau$$

we conclude that $\mathcal{D}(T)$ is dense in V. We now prove that T is closed. Let us take a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ such that $\lim_{n\to\infty} v_n = v_0$ and $\lim_{n\to\infty} Tv_n = y_0$. Thanks to Proposition 3.8 point (3) we have

$$\mathcal{U}(t)v_0 - v_0 = \lim_{n\to\infty}\big[\mathcal{U}(t)v_n - v_n\big] = \lim_{n\to\infty}\int_0^t \mathcal{U}(\tau)Tv_n\,d\tau = \int_0^t \mathcal{U}(\tau)y_0\,d\tau$$

where we have used the fact that

$$\lim_{n\to\infty} \mathcal{U}(\tau)Tv_n = \mathcal{U}(\tau)y_0, \quad \text{uniformly in } [0,t].$$

Proposition 3.8 point (4) yields

$$\lim_{t\to 0^+} T_t v_0 = \lim_{t\to 0^+} \frac{1}{t}\int_0^t \mathcal{U}(\tau)y_0\,d\tau = y_0.$$

This implies that $v_0 \in \mathcal{D}(T)$ and $Tv_0 = y_0$, i.e. T is closed. □

We can now prove the following characterization result.

Theorem 3.10. *Let $\mathcal{U}(t)$ be a quaternionic semigroup on a quaternionic Banach space V. Then $\mathcal{U}(t)$ has a bounded infinitesimal quaternionic generator if and only if it is uniformly continuous.*

Proof. If $\mathcal{U}(t)$ is a uniformly continuous semigroup then by Theorem 3.2 it has a bounded infinitesimal quaternionic generator. To prove the converse, we suppose that $\mathcal{U}(t)$ has a bounded infinitesimal quaternionic generator T. Lemma 3.9 implies that T is defined everywhere. By Proposition 3.4 we have that for every $\tau \geqslant 0$ there exists a positive constant $C(\tau)$ such that

$$\big\|\mathcal{U}(t)\big\| \leqslant C(\tau), \quad \text{for } \tau \geqslant 0,\ |t-\tau| \leqslant 1.$$

The semigroup properties give

$$\mathcal{U}(t) - \mathcal{U}(\tau) = \mathcal{U}(\tau)\big[\mathcal{U}(t-\tau) - \mathcal{I}\big] = (t-\tau)\mathcal{U}(\tau)T_{t-\tau}, \quad \text{for } t > \tau, \tag{19}$$

and

$$\mathcal{U}(t) - \mathcal{U}(\tau) = -\mathcal{U}(t)\big[\mathcal{U}(\tau-t) - \mathcal{I}\big] = -(\tau-t)\mathcal{U}(t)T_{\tau-t}, \quad \text{for } \tau > t, \tag{20}$$

where $T_{\tau-t}$ and $T_{t-\tau}$ are as in Definition 3.7. Proposition 3.4 and the Principle of Uniform Boundedness imply

$$\sup_{\tau>t|\tau-t\leqslant 1} \|T_{\tau-t}\| = K < +\infty.$$

By taking the norm of (19) and (20), we get

$$\left\|\mathcal{U}(t)-\mathcal{U}(\tau)\right\| \leqslant C(\tau)K|t-\tau|, \quad \text{for } t \geqslant 0,\ |t-\tau| \leqslant 1,$$

which proves that $\mathcal{U}(t)$ is a uniformly continuous quaternionic semigroup. $\square$

3.1. Some comments

Since the exponential function is both left and right regular, point (3) in Theorem 3.2 follows also from the right version of the quaternionic functional calculus.

Theorem 3.11. *Let the assumptions of Theorem* 3.2 *hold. Then we have*

$$\frac{d}{dt}e^{tT} = Te^{tT} = e^{tT}T.$$

Proof. Let

$$e^{tT} = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} e^{ts}\, ds_I\, S_R^{-1}(s,T)$$

where U is as in Definition 2.9 and contains the S-spectrum of the bounded operator T. Note that the S-spectrum is a closed and bounded set in $\mathbb{H}$ thanks to Theorem 5.4 in [6]. Let $h>0$ and consider

$$\frac{e^{(t+h)T}-e^{tT}}{h} = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \frac{(e^{(t+h)s}-e^{ts})s^{-1}}{h} s\, ds_I\, S_R^{-1}(s,T).$$

By taking the limit for $h\to 0$ we get

$$\frac{d}{dt}e^{tT} = \lim_{h\to 0}\frac{e^{(t+h)T}-e^{tT}}{h} = e^{tT}T.$$

Finally, formula

$$\frac{(e^{(t+h)s}-e^{ts})s^{-1}}{h}s = s\frac{s^{-1}(e^{(t+h)s}-e^{ts})}{h}$$

yields

$$e^{tT}T = Te^{tT}. \qquad \square$$

Let us introduce the Laplace transform that gives the right S-resolvent operator.

Theorem 3.12. *Let $T \in \mathcal{B}(V)$ and let $s_0 > \|T\|$. Then the right S-resolvent operator $S_R^{-1}(s,T)$ is given by*

$$S_R^{-1}(s,T) = \int_0^{+\infty} e^{-ts} e^{tT}\, dt. \tag{21}$$

Proof. Consider, for $\bar{s} \notin \sigma_L(T)$,

$$e^{-ts} e^{tT} S_R(s,T) = -e^{-ts} e^{tT} \big(T^2 - 2s_0 T + |s|^2 \mathcal{I}\big)(T - \bar{s}\mathcal{I})^{-1},$$

since T and e^{tT} commute we can write

$$\begin{aligned} e^{-ts} e^{tT} S_R(s,T) &= -e^{-ts} \big(T^2 - 2s_0 T + |s|^2 \mathcal{I}\big) e^{tT} (T - \bar{s}\mathcal{I})^{-1} \\ &= -\frac{d}{dt}\big[e^{-ts}(T - \bar{s}\mathcal{I}) e^{tT}\big](T - \bar{s}\mathcal{I})^{-1} \\ &= -\frac{d}{dt}\big[e^{-ts}(T - \bar{s}\mathcal{I}) e^{tT}(T - \bar{s}\mathcal{I})^{-1}\big]. \end{aligned}$$

For $\theta > 0$ we have

$$\begin{aligned} \int_0^{\theta} e^{-ts} e^{tT} S_R(s,T)\, dt &= -\int_0^{\theta} \frac{d}{dt}\big[e^{-ts}(T - \bar{s}\mathcal{I}) e^{tT}(T - \bar{s}\mathcal{I})^{-1}\big]\, dt \\ &= \mathcal{I} - e^{-\theta s}(T - \bar{s}\mathcal{I}) e^{\theta T}(T - \bar{s}\mathcal{I})^{-1}. \end{aligned}$$

Since we have assumed that $s_0 > \|T\|$, for $\theta \to +\infty$, we get

$$\big\|e^{-\theta s}(T - \bar{s}\mathcal{I}) e^{\theta T}(T - \bar{s}\mathcal{I})^{-1}\big\| \leqslant e^{-\theta s_0} e^{\theta \|T\|} \big\|(T - \bar{s}\mathcal{I})\big\| \big\|(T - \bar{s}\mathcal{I})^{-1}\big\| \to 0$$

so the statement follows. □

4. Strongly continuous quaternionic semigroup

We now treat the case of strongly continuous quaternionic semigroup. We show that, in a suitable sense, it is possible to define the Laplace transform of a quaternionic semigroup. Moreover we prove that the Hille–Yosida–Phillips theorem can be extended to the quaternionic setting.

Theorem 4.1. *Let $\{\mathcal{U}(t)\}_{t \geqslant 0}$ be a strongly continuous quaternionic semigroup and let T_h be as in Definition 3.7. Then*

$$\mathcal{U}(t)v = \lim_{h \to 0^+} e^{tT_h} v, \quad v \in V,$$

uniformly for t in any finite interval.

Proof. First of all, we observe that since t and h are real numbers the operators $\frac{t}{h}\mathcal{I}$ and $\frac{t}{h}\mathcal{U}(h)$ commute, so the quaternionic operator e^{tT_h} can be written as

$$e^{tT_h} = e^{-\frac{t}{h}\mathcal{I}} e^{\frac{t}{h}\mathcal{U}(h)}.$$

Since $\mathcal{U}$ is a bounded operator we can use its power series expansion and, for the semigroup properties, we have

$$e^{tT_h} = e^{-\frac{t}{h}\mathcal{I}} \sum_{n\geqslant 0} \frac{t^n}{n!h^n}\mathcal{U}(nh). \tag{22}$$

By taking the norm of (22) and using point (2) in Proposition 3.6, for every $\delta > \omega_0$ there exists a positive constant M_δ such that

$$\left\| e^{tT_h} \right\| \leqslant e^{-\frac{t}{h}} M_\delta \sum_{n\geqslant 0} \frac{t^n}{n!h^n} e^{nh\delta} = M_\delta e^{m(t,\delta,h)},$$

where

$$m(t,\delta,h) := \frac{t}{h}\big(e^{h\delta} - 1\big).$$

So there exists a positive constant K_t for which

$$\left\| e^{\tau T_h} \right\| \leqslant K_t, \quad \text{for } \tau \in [0,t],\ h \in (0,1].$$

Now if $v \in \mathcal{D}(T)$ and $t \leqslant t_0$ then, by point (3) in Proposition 3.8, we obtain

$$\mathcal{U}(t)v - e^{tT_h}v = \int_0^t \frac{d}{d\tau}\big(e^{(t-\tau)T_h}\mathcal{U}(\tau)v\big)\,d\tau = \int_0^t e^{(t-\tau)T_h}\mathcal{U}(\tau)(Tv - T_h v)\,d\tau.$$

Taking the norm we have

$$\left\| \mathcal{U}(t)v - e^{tT_h}v \right\| \leqslant t_0 K_{t_0} M_\delta e^{\delta t_0} \|Tv - T_h v\|,$$

so

$$\left\| \mathcal{U}(t)v - e^{tT_h}v \right\| \to 0 \quad \text{as } h \to 0.$$

Lemma 3.9 implies that $\mathcal{D}(T)$ is dense in V. The statement follows by Theorem II.3.6 in [18] which holds also in the quaternionic setting, with obvious modifications. □

Now we introduce the Laplace transform for strongly continuous quaternionic semigroups.

Theorem 4.2. *Let $\mathcal{U}(t)$ be a strongly continuous quaternionic semigroup and set*

$$\omega_0 := \lim_{t\to+\infty} \frac{1}{t} \ln\left\| \mathcal{U}(t) \right\|.$$

Assume that $\mathcal{U}(t)$ is generated by a linear quaternionic operator T and take $s \in \mathbb{H}$ such that $\mathrm{Re}[s] > \omega_0$. *Then we have that $s \in \rho_S(T)$ and the extended left S-resolvent operator is given by*

$$\widehat{S}_L^{-1}(s,T)v = \int_0^\infty \mathcal{U}(t)e^{-ts}v\,dt, \quad v \in V. \tag{23}$$

Proof. Assume that $\omega_0 < \delta < \mathrm{Re}[s]$. By Proposition 3.6 there exists a positive constant M_δ such that

$$\|\mathcal{U}(t)\| \leqslant M_\delta e^{\delta t}, \quad \text{for } t \geqslant 0. \tag{24}$$

As a consequence, the integral (23) exists for $\mathrm{Re}[s] > \omega_0$ and defines a bounded linear operator. Let us set, for $\mathrm{Re}[s] > \omega_0$:

$$F(s)v := \int_0^\infty \mathcal{U}(t)e^{-ts}v\,dt, \quad v \in V.$$

We have to prove that $F(s)v = \widehat{S}_L^{-1}(s,T)v$ for $v \in V$. Let us begin by proving that $F(s)v \in \mathcal{D}(T)$. Apply the operator T_h to $F(s)v$ and observe that

$$\begin{aligned}(T_hF)(s)v &= \int_0^\infty \frac{\mathcal{U}(h)-\mathcal{I}}{h}\mathcal{U}(t)e^{-ts}v\,dt \\ &= \frac{1}{h}\int_0^\infty \mathcal{U}(t+h)e^{-ts}v\,dt - \frac{1}{h}\int_0^\infty \mathcal{U}(t)e^{-ts}v\,dt.\end{aligned}$$

With a change of variable we get

$$(T_hF)(s)v = \frac{1}{h}\int_h^\infty \mathcal{U}(\tau)e^{-s(\tau-h)}v\,d\tau - \frac{1}{h}\int_0^\infty \mathcal{U}(t)e^{-ts}v\,dt.$$

By denoting again the variable τ by t we have

$$\begin{aligned}(T_hF)(s)v &= \frac{1}{h}\int_0^h \mathcal{U}(t)e^{-s(t-h)}v\,dt + \frac{1}{h}\int_h^\infty \mathcal{U}(t)e^{-s(t-h)}v\,dt \\ &\quad - \frac{1}{h}\int_0^\infty \mathcal{U}(t)e^{-ts}v\,dt - \frac{1}{h}\int_0^h \mathcal{U}(t)e^{-s(t-h)}v\,dt \\ &= \int_0^\infty \mathcal{U}(t)e^{-st}\frac{(e^{sh}-1)}{h}v\,dt - \frac{1}{h}\int_0^h \mathcal{U}(t)e^{-s(t-h)}v\,dt.\end{aligned}$$

So we finally obtain

$$T_h F(s)v = F(s)\frac{1}{h}\left(e^{sh}-1\right)v - \frac{1}{h}\int_0^h \mathcal{U}(t)e^{-st}e^{sh}v\,dt. \tag{25}$$

Taking the limit for $h \to 0$ in (25), and using point (4) in Proposition 3.8, we obtain

$$TF(s)v = F(s)sv - \mathcal{I}v, \quad \text{for every } v \in V. \tag{26}$$

This fact implies that $F(s)v \in \mathcal{D}(T)$ and it satisfies the extended S-resolvent equation (15). Let us prove that

$$S_L(s,T)F(s)v = \mathcal{I}v, \quad \text{for all } v \in V, \tag{27}$$

and also

$$F(s)S_L(s,T)v = \mathcal{I}v, \quad \text{for all } v \in \mathcal{D}(T), \tag{28}$$

which implies that $F(s)v = \widehat{S}_L^{-1}(s,T)v$ for every $v \in V$ (see [10, Theorem 5.12]). So we verify that

$$\left[(T-\bar{s}\mathcal{I})^{-1}s(T-\bar{s}\mathcal{I}) - T\right]F(s)v = \mathcal{I}v, \quad \text{for all } v \in V.$$

We have

$$(T-\bar{s}\mathcal{I})^{-1}\left[sTF(s)v - |s|^2F(s)v\right] = TF(s)v + \mathcal{I}v, \quad \text{for all } v \in V,$$

and using (26) we obtain

$$(T-\bar{s}\mathcal{I})^{-1}\left[sF(s)sv - sv - |s|^2F(s)v\right] = F(s)sv, \quad \text{for all } v \in V,$$
$$(T-\bar{s}\mathcal{I})^{-1}\left[sF(s)sv - sv - F(s)\bar{s}sv\right] = F(s)sv, \quad \text{for all } v \in V,$$

and

$$(T-\bar{s}\mathcal{I})^{-1}\left[sF(s) - \mathcal{I} - F(s)\bar{s}\right]sv = F(s)sv, \quad \text{for all } v \in V.$$

We now observe that (26) can be written as $\mathcal{I}v = F(s)sv - TF(s)v$, so we can write

$$(T-\bar{s}\mathcal{I})^{-1}\left[sF(s) + TF(s) - F(s)2\operatorname{Re}(s)\right]sv = F(s)sv, \quad \text{for all } v \in V,$$

from which we get

$$(T-\bar{s}\mathcal{I})^{-1}\left[TF(s) - \bar{s}F(s)\right]sv = F(s)sv, \quad \text{for all } v \in V,$$

that is

$$(T-\bar{s}\mathcal{I})^{-1}(T-\bar{s}\mathcal{I})F(s)(sv) = F(s)(sv), \quad \text{for all } v \in V,$$

which proves (27). To verify (28) observe that from (27) we get

$$\big(F(s)S_L(s,T)\big)F(s)v = F(s)v, \quad \text{for all } v \in V,$$

but since $F(s)v \in \mathcal{D}(T)$ for $v \in V$ we have that $F(s)S_L(s,T)w = \mathcal{I}w$ for all $w \in \mathcal{D}(T)$. □

Let T be a linear quaternionic operator, $\bar{s} \in \mathbb{H}$ and let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Let us introduce the following notation:

$$(T - \bar{s}\mathcal{I})^{n*} := \sum_{k=0}^{n} \binom{n}{k} T^{n-k} (\bar{s})^k, \quad k, n \in \mathbb{N}.$$

Let $Q_s(T)$ be as in Definition 2.14 and let $s \in \rho_S(T)$. The operator $Q_s(T)^n(T - \bar{s}\mathcal{I})^{n*}$ is defined on $\mathcal{D}(T^n)$ and admits a continuous extension on V denoted by the symbol "$\widehat{\ }$", i.e.

$$Q_s(T)\widehat{^n(T - \bar{s}}\mathcal{I})^{n*} = \sum_{n=0}^{n} \binom{n}{k} T^{n-k} Q_s(T)^n (\bar{s})^k. \tag{29}$$

Theorem 4.3 *(Hille–Yosida–Phillips: necessary condition). Let T be a closed linear quaternionic operator with dense domain whose S-spectrum lies in the half space* $\mathrm{Re}[s] \leqslant \omega$, *where $\omega \in \mathbb{R}$. Let $\mathcal{U}(t)$ be a strongly continuous semigroup for $t \geqslant 0$ and assume that there exist $M > 0$ and $\omega \in \mathbb{R}$ such that:*

$$\big\|\mathcal{U}(t)\big\| \leqslant M e^{\omega t}, \quad t \geqslant 0,$$

and

$$\widehat{S}_L^{-1}(s,T)v = \int_0^\infty \mathcal{U}(t) e^{-st} v\, dt, \quad v \in V. \tag{30}$$

Then we have the following estimate

$$\big\| Q_s(T)\widehat{^n(T - \bar{s}}\mathcal{I})^{n*} \big\| \leqslant \frac{M}{(\mathrm{Re}[s] - \omega)^n}, \quad n \in \mathbb{N}. \tag{31}$$

Proof. Recall that for slice regular functions the derivative $\frac{d}{ds}$ equals the partial derivative with respect to the real part of s and that the function $S_L^{-1}(s,T)$ is right regular in the variable s. Simple computations, for $s \in \rho_S(T)$, yield

$$\frac{d}{ds} S_L^{-1}(s,T) = -Q_s(T)^2\big(T^2 - 2T\bar{s} + \mathcal{I}(\bar{s})^2\big), \quad \text{on } \mathcal{D}\big(T^2\big),$$

and, by induction,

$$\frac{d^n}{ds^n} S_L^{-1}(s,T) = (-1)^n n! Q_s(T)^{1+n}(T - \bar{s}\mathcal{I})^{(n+1)*}, \quad \text{on } \mathcal{D}\big(T^{n+1}\big). \tag{32}$$

Observe that for every $n \in \mathbb{N}$ the operator (32) admits a continuous extension on V defined by

$$\widehat{\frac{d^n}{ds^n} S_L^{-1}}(s,T) = (-1)^n n! \sum_{n=0}^{n} \binom{n}{k} T^{n-k} Q_s(T)^{1+n} (\bar{s})^k, \quad s \in \rho_S(T). \tag{33}$$

Now consider the integral

$$\int_0^\infty \mathcal{U}(t) e^{-st} t^n v\, dt, \quad n \in \mathbb{N},$$

and observe that it is possible to take the derivative with respect to s under the integral, so that

$$\frac{d}{ds} \int_0^\infty \mathcal{U}(t) e^{-st} t^n v\, dt = -\int_0^\infty \mathcal{U}(t) e^{-st} t^{n+1} v\, dt, \quad \text{for } \mathrm{Re}[s] > \omega,$$

and, inductively,

$$\frac{d^n}{ds^n} \int_0^\infty \mathcal{U}(t) e^{-st} v\, dt = \int_0^\infty \mathcal{U}(t) e^{-st} (-t)^n v\, dt, \quad \text{for } \mathrm{Re}[s] > \omega.$$

Since we have assumed that (30) holds we also have

$$\widehat{\frac{d^n}{ds^n} S_L^{-1}}(s,T) v = \int_0^\infty \mathcal{U}(t) e^{-st} (-t)^n v\, dt, \quad \text{for } \mathrm{Re}[s] > \omega. \tag{34}$$

From (32), (33) and (34) we get

$$Q_s(T)^{n} \widehat{(T - \bar{s}\mathcal{I})^{n*}} v = \frac{1}{(n-1)!} \int_0^\infty \mathcal{U}(t) e^{-st} t^{n-1} v\, dt;$$

and, by taking the norm, we have the estimate

$$\left\| Q_s(T)^{n} \widehat{(T - \bar{s}\mathcal{I})^{n*}} \right\| \leqslant \frac{M}{(n-1)!} \int_0^\infty e^{-t(\mathrm{Re}[s]-\omega)} t^{n-1}\, dt.$$

Recalling that $\int_0^\infty e^{-\tau} \tau^{n-1}\, d\tau = (n-1)!$, it easily follows the equality

$$\int_0^\infty e^{-t(\mathrm{Re}[s]-\omega)} t^{n-1}\, dt = \frac{(n-1)!}{(\mathrm{Re}[s]-\omega)^n}$$

from which (31) follows. $\square$

Let V_0 be a subset of V and let $\operatorname{span}(V_0)$ be the subspace of V spanned by V_0. We say that V_0 is a fundamental set if $\operatorname{span}(V_0) = V$. This definition is useful to state the following result.

Theorem 4.4. *Let V be a quaternionic Banach space and let $\mathcal{A}_m$ be a sequence of linear bounded quaternionic operators on V to itself. Then the limit $\mathcal{A}v = \lim_{m\to\infty} \mathcal{A}_m v$ exists for every $v \in V$ if and only if*

(a) *the limit $\mathcal{A}v$ exists for every fundamental set,*
(b) *for each $v \in V$ we have $\sup_{m\in\mathbb{N}} \|\mathcal{A}_m v\| < \infty$.*

When the limit $\mathcal{A}v$ exists for each $v \in V$, the operator $\mathcal{A}$ is bounded and

$$\|\mathcal{A}\| \leqslant \liminf_{m\to\infty} \|\mathcal{A}_m\| \leqslant \sup_{m\in\mathbb{N}} \|\mathcal{A}_m\| < \infty.$$

Proof. It mimics the proof of Theorem II.3.6 in [18] for complex Banach spaces. □

Definition 4.5 *(Yosida approximations).* Let $T \in \mathcal{K}(V)$ and $s \in \rho_S(T)$. We define the left and right Yosida approximations of T as

$$\mathcal{Y}_L(s) := -\big[\mathcal{I} - S_L^{-1}(s,T)s\big]s \tag{35}$$

and

$$\mathcal{Y}_R(s) := -s\big[\mathcal{I} - sS_R^{-1}(s,T)\big] \tag{36}$$

respectively.

Remark 4.6. When $s \in \mathbb{R}$ we obviously have $\mathcal{Y}_L(s) = \mathcal{Y}_R(s)$.

Lemma 4.7. *Let $T \in \mathcal{K}(V)$, $s_0 \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and assume that the S-spectrum lies in the half space $s_0 \leqslant \omega$, where $\omega \in \mathbb{R}$. Then we have*

$$\lim_{s_0\to\infty} \mathcal{Y}_L(s_0)v = Tv, \quad v \in \mathcal{D}(T). \tag{37}$$

Proof. Since s_0 is a real number such that $s_0 > \omega$, and $s_0 \in \rho_S(A) \cap \mathbb{R}$, we have

$$\mathcal{Y}_L(s_0)v = -\big[\mathcal{I} - S_L^{-1}(s_0,T)s_0\big]s_0 v = -\big[\mathcal{I} - s_0(s_0\mathcal{I} - T)^{-1}\big]s_0 v = s_0(s_0\mathcal{I} - T)^{-1}Tv$$

for $v \in \mathcal{D}(T)$. Set, for $w \in V$,

$$\mathcal{A}_{s_0} w := s_0(s_0\mathcal{I} - T)^{-1} w$$

and observe that

$$\|\mathcal{A}_{s_0} w\| = \big\|s_0(s_0\mathcal{I} - T)^{-1} w\big\| \leqslant \frac{s_0 M}{s_0 - \omega} \leqslant 2M, \quad \text{for } s_0 \text{ large.}$$

So by Theorem 4.4 we have that $\mathcal{A}_{s_0}$ converges to a bounded linear operator:

$$\lim_{s_0\to\infty} s_0(s_0\mathcal{I}-T)^{-1}w = w.$$

Now consider

$$\mathcal{Y}_L(s_0)v - Tv = s_0(s_0\mathcal{I}-T)^{-1}Tv - Tv,$$

and taking the limit for $s_0\to\infty$ we obtain $\|\mathcal{Y}_L(s_0)v - Tv\|\to 0$ for all $v\in\mathcal{D}(T)$. □

Theorem 4.8 *(Hille–Yosida–Phillips: sufficient condition). If there exist $M>0$ and $\omega\in\mathbb{R}$ such that for every real number $s_0>\omega$, with $s_0\in\rho_S(T)$, we have*

$$\left\|(s_0\mathcal{I}-T)^{-n}\right\| \leqslant \frac{M}{(s_0-\omega)^n}, \quad n\in\mathbb{N},$$

then the closed linear quaternionic operator T, with dense domain, is the infinitesimal generator of a strongly continuous semigroup.

Proof. Since $s_0\in\rho_S(T)$ and $s_0>\omega$, then

$$\mathcal{Y}_L(s_0) = -s_0\mathcal{I} + s_0^2(s_0\mathcal{I}-T)^{-1},$$

and the bounded operators $s_0\mathcal{I}$ and $s_0^2(s_0\mathcal{I}-T)^{-1}$ commute. So we can write

$$e^{-t\mathcal{Y}_L(s_0)} = e^{-ts_0}e^{ts_0^2(s_0\mathcal{I}-T)^{-1}} = e^{-ts_0}\sum_{n\geqslant 0}\frac{1}{n!}\left(ts_0^2\right)^n(s_0\mathcal{I}-T)^{-n}.$$

Taking the norm we have

$$\begin{aligned}\left\|e^{t\mathcal{Y}_L(s_0)}\right\| &\leqslant e^{-ts_0}\sum_{n\geqslant 0}\frac{1}{n!}\left(ts_0^2\right)^n\left\|(s_0\mathcal{I}-T)^{-n}\right\| \\ &\leqslant Me^{-ts_0}\sum_{n\geqslant 0}\frac{(ts_0^2)^n}{n!(s_0-\omega)^n} = M\exp(-ts_0)\exp\left(\frac{ts_0^2}{s_0-\omega}\right) = M\exp\left(\frac{ts_0\omega}{s_0-\omega}\right).\end{aligned}$$

If $\omega_0>\omega$, then for s_0 sufficiently large we have

$$\frac{s_0\omega}{s_0-\omega} < \omega_0,$$

and for all $t\geqslant 0$ we obtain

$$\left\|e^{t\mathcal{Y}_L(s_0)}\right\| \leqslant Me^{t\omega_0}. \tag{38}$$

Lemma 4.7 implies that $\lim_{s_0\to\infty}\mathcal{Y}_L(s_0)v = Tv$, for $v\in\mathcal{D}(T)$. Observe that for any $s_0, p_0\in\mathbb{R}$ we have

$$\mathcal{Y}_L(s_0)\mathcal{Y}_L(p_0) = \big[-s_0\mathcal{I} + s_0^2(s_0\mathcal{I}-T)^{-1}\big]\big[-p_0\mathcal{I} + p_0^2(p_0\mathcal{I}-T)^{-1}\big] = \mathcal{Y}_L(p_0)\mathcal{Y}_L(s_0).$$

Let us set

$$\mathcal{U}_{s_0}(t) := e^{t\mathcal{Y}_L(s_0)}.$$

If we use the power series expansion

$$\mathcal{U}_{s_0}(t) := \sum_{n\geqslant 0}\frac{1}{n!}t^n\big(\mathcal{Y}_L(s_0)\big)^n$$

we see that

$$\mathcal{Y}_L(p_0)\mathcal{U}_{s_0}(t) := \sum_{n\geqslant 0}\frac{1}{n!}t^n\big(\mathcal{Y}_L(s_0)\big)^n\mathcal{Y}_L(p_0) = \mathcal{U}_{s_0}(t)\mathcal{Y}_L(p_0).$$

Take $v\in\mathcal{D}(T)$ and apply point (3) in Proposition 3.8 to get

$$\begin{aligned}
\mathcal{U}_{s_0}(t)v - \mathcal{U}_{p_0}(t)v &= \int_0^t \frac{d}{d\tau}\big[\mathcal{U}_{p_0}(t-\tau)\mathcal{U}_{s_0}(\tau)v\big]\,d\tau \\
&= \int_0^t \mathcal{U}_{p_0}(t-\tau)\big[\mathcal{Y}_L(s_0) - \mathcal{Y}_L(p_0)\big]\mathcal{U}_{s_0}(\tau)v\,d\tau \\
&= \int_0^t \mathcal{U}_{p_0}(t-\tau)\mathcal{U}_{s_0}(\tau)\big[\mathcal{Y}_L(s_0) - \mathcal{Y}_L(p_0)\big]v\,d\tau.
\end{aligned}$$

By taking the norm and considering s_0 and p_0 sufficiently large, we have

$$\big\|\mathcal{U}_{s_0}(t)v - \mathcal{U}_{p_0}(t)v\big\| \leqslant M^2te^{t\omega_0}\big\|\big[\mathcal{Y}_L(s_0) - \mathcal{Y}_L(p_0)\big]v\big\|.$$

So $\mathcal{U}_{s_0}(t)v$ converges to a limit uniformly in each finite interval. By assumption $\mathcal{D}(T)$ is dense in V. Thanks to estimate (38) and to Theorem 4.4, there exists a linear quaternionic bounded operator $\mathcal{U}(t)$ such that

$$\lim_{s_0\to\infty}\mathcal{U}_{s_0}(t)v = \mathcal{U}(t)v, \qquad v\in V,$$

moreover

$$\big\|\mathcal{U}(t)v\big\| \leqslant \liminf_{s_0\to\infty}\big\|\mathcal{U}_{s_0}(t)v\big\| \leqslant e^{t\omega_0}.$$

We observe that the map $t \to \mathcal{U}(t)v$ is continuous because of the uniform convergence. The fact that $\mathcal{U}(t)$ is a semigroup follows from the fact that $\mathcal{U}_{s_0}(t)$ is a semigroup. Now observe that the following estimates hold

$$\begin{aligned}\big\|\mathcal{U}_{s_0}(t)\mathcal{Y}_L(s_0)v - \mathcal{U}(t)Tv\big\| &\leqslant \big\|\mathcal{U}_{s_0}(t)\big[\mathcal{Y}_L(s_0)v - Tv\big]\big\| + \big\|\big[\mathcal{U}_{s_0}(t) - \mathcal{U}(t)\big]Tv\big\| \\ &\leqslant Me^{t\omega_0}\big\|\mathcal{Y}_L(s_0)v - Tv\big\| + 2Me^{t\omega_0}\|Tv\| \end{aligned} \tag{39}$$

for any $v \in \mathcal{D}(T)$. So by estimate (39) and the Lebesgue Dominated Convergence Theorem, we can take the limit as $s_0 \to \infty$ in the both sides of

$$\mathcal{U}_{s_0}(t)v - v = \int_0^t \mathcal{U}_{s_0}(\tau)\mathcal{Y}_L(s_0)v\,d\tau$$

and we get

$$\mathcal{U}(t)v - v = \int_0^t \mathcal{U}(\tau)Tv\,d\tau.$$

Now if we denote by Z the infinitesimal quaternionic generator of $\mathcal{U}(t)$ we have

$$Zv = \lim_{t\to 0}\frac{\mathcal{U}(t)v - v}{t} = \lim_{t\to 0}\frac{1}{t}\int_0^t \mathcal{U}(\tau)Tv\,d\tau = Tv, \quad \text{for } v \in \mathcal{D}(T).$$

This means that $\mathcal{D}(T) \subseteq \mathcal{D}(Z)$ and Z is an extension of T. But for s_0 large, $s_0 \in \rho_S(T) \cap \rho_S(Z)$, and recalling that $S(s_0, T) = T - s_0\mathcal{I}$, we get the relations

$$(T - s_0\mathcal{I})\mathcal{D}(T) = V, \qquad (Z - s_0\mathcal{I})\mathcal{D}(T) = V, \qquad (Z - s_0\mathcal{I})\mathcal{D}(Z) = V,$$

which imply $\mathcal{D}(T) = \mathcal{D}(Z)$, and so $Z = T$. □

There are several consequences of the Hille–Yosida–Phillips theorem.

Corollary 4.9. *Let T be a linear closed quaternionic operator with dense domain. Then T generates a strongly continuous quaternionic semigroup $\mathcal{U}(t)$ of bounded quaternionic operators if and only if for some real number ω such that*

$$\big\|\mathcal{U}(t)\big\| \leqslant e^{t\omega}$$

we have

$$\big\|S_L^{-1}(s_0, T)\big\| \leqslant \frac{1}{s_0 - \omega}, \quad s_0 > \omega. \tag{40}$$

Proof. The Laplace transform of the semigroup

$$S_L^{-1}(s_0, T) = \int_0^\infty \mathcal{U}(t) e^{-s_0 t} v \, dt, \quad v \in V,$$

and Theorem 4.3 imply the necessity of the estimate (40). Now consider $s \in \rho_S(T) \cap \mathbb{R}$. We have

$$S_L^{-1}(s_0, T) = (s_0 \mathcal{I} - T)^{-1}.$$

Estimate (40) implies that

$$\left\| S_L^{-1}(s_0, T)^n \right\| \leqslant \left\| S_L^{-1}(s_0, T) \right\|^n \leqslant \frac{1}{(s_0 - \omega)^n}, \quad s_0 > \omega,$$

and by Theorem 4.8 the operator T is the quaternionic generator of a semigroup $\mathcal{U}(t)$. From the proof of Theorem 4.8 we also have that $\|\mathcal{U}(t)\| \leqslant e^{t\omega}$, for $M = 1$. □

To prove the next corollary we need a technical lemma. The result is well known for real functions (see for example VIII.1.15 in [18]). The proof of the quaternionic version follows the same lines.

Lemma 4.10. *Let $u \in L^1(0, \infty; \mathbb{H})$, $s \in \mathbb{H}$ and*

$$\int_0^\infty e^{-st} u(t) \, dt = 0$$

for $\mathrm{Re}[s]$ *sufficiently large. Then $u(t) = 0$ almost everywhere.*

Corollary 4.11. *Let T be a linear closed quaternionic operator with dense domain. Then T is the quaternionic infinitesimal generator of a strongly continuous quaternionic semigroup if and only if there exists a strongly continuous family $\mathcal{W}(t)$, $t \geqslant 0$, of bounded linear quaternionic operators satisfying, for some real numbers $M > 0$ and ω, the conditions:*

- $\mathcal{W}(0) = \mathcal{I}$,
- $\|\mathcal{W}(t)\| \leqslant M e^{t\omega}$,
- $S_R^{-1}(s_0, T) = \int_0^\infty e^{-s_0 t} \mathcal{W}(t) v \, dt$, $s_0 > \omega$.

Moreover, $\mathcal{W}(t)$ is the quaternionic semigroup with infinitesimal generator T.

Proof. Theorem 4.3 implies

$$\left\| (T - s_0 \mathcal{I})^{-n} \right\| \leqslant \frac{M}{(s_0 - \omega)^n}, \quad \text{for } s_0 > \omega,\ n \in \mathbb{N},$$

and by Theorem 4.8 operator T is the infinitesimal generator of a semigroup $\mathcal{U}(t)$ and $\|\mathcal{U}(t)\| \leqslant Me^{t\omega}$. Theorem 4.2 yields

$$S_R^{-1}(s_0, T)v = (T - s_0\mathcal{I})^{-1}v = \int_0^\infty e^{-ts_0}\mathcal{U}(t)v\,dt, \quad v \in V,\ s_0 > \omega.$$

We now reason by duality. Let φ be an element in the dual space V',

$$\int_0^\infty e^{-s_0 t}\big\langle\varphi, \mathcal{U}(t)v - \mathcal{W}(t)v\big\rangle\,dt = 0, \quad \text{for } s_0 > \omega,$$

and define the function

$$u(t) := e^{-(\omega+1)t}\big\langle\varphi, \mathcal{U}(t)v - \mathcal{W}(t)v\big\rangle, \quad \varphi \in V'.$$

We have

$$\int_0^\infty e^{-s_0 t}u(t)\,dt = 0, \quad \text{for } s_0 \geqslant 0.$$

From Lemma 4.10 we get that $\langle\varphi, \mathcal{U}(t)v - \mathcal{W}(t)v\rangle = 0$ for almost all $t \geqslant 0$. By continuity such equation holds for all $t \geqslant 0$ and thus as a consequence of the quaternionic version of the Hahn–Banach Theorem, we get $\mathcal{U}(t) = \mathcal{W}(t)$ for all $t \geqslant 0$. □

Remark 4.12. The solution of the problem of determining which unbounded closed operators are infinitesimal generators of strongly continuous semigroups allows us to consider the abstract Cauchy problem: *given a closed unbounded quaternionic operator T, determine a quaternionic function $q(t)$, defined for $t \geqslant 0$, such that $q(t)$ belongs to the domain of T and that satisfies the problem*

$$\frac{dq}{dt} = Tq(t), \qquad q(0) = q_0,$$

where q_0 is a given datum.

5. Strongly continuous groups of quaternionic operators

Now we consider the problem to characterize *when a strongly continuous semigroup of operators defined on* $[0, \infty)$ *can be extended to a group of operators defined on* $\mathbb{R}$. This extension is unique if it exists and if the family $\mathcal{Z}(t) = \mathcal{U}(-t)$, for $t \geqslant 0$, is a strongly continuous semigroup. Consider the identity

$$\frac{1}{h}\big[\mathcal{Z}(t)v - v\big] = \frac{1}{-h}\big[-\mathcal{U}(-2)\big[\mathcal{U}(2-t)v - \mathcal{U}(2)v\big]\big], \quad \text{for } t \in (0, 1).$$

By taking the limit for $h \to 0$ we have that the infinitesimal generator is $-T$ and $\mathcal{D}(-T) = \mathcal{D}(T)$. In this case T is called the quaternionic infinitesimal generator of the group $\mathcal{Z}(t)$. Let us prove a necessary and sufficient condition such that the semigroup can be extended to a group.

Theorem 5.1. *Let T be a linear closed quaternionic operator with dense domain. Then T is the quaternionic infinitesimal generator of a strongly continuous quaternionic group of bounded operators if and only if there exist real numbers $M > 0$ and $\omega \geqslant 0$ such that*

$$\left\| \left(S_L^{-1}(s_0, T)\right)^n \right\| \leqslant \frac{M}{(|s_0| - \omega)^n}, \quad s_0 > \omega \text{ and } s_0 < -\omega. \tag{41}$$

If T generates the group $\{\mathcal{Z}(t)\}_{t\in\mathbb{R}}$, then $\|\mathcal{Z}(t)\| \leqslant Me^{\omega|t|}$.

Proof. The necessity of estimate (41) follows from the above considerations, from Theorem 4.8 and the relation

$$S_L^{-1}(s_0, -T) = (s_0\mathcal{I} - T)^{-1} = -(-s_0\mathcal{I} - T)^{-1} = S_L^{-1}(-s_0, T),$$

since s_0 is a real number and $\sigma_S(-T) = -\sigma_S(T)$. This last relation between the S-spectra of $-T$ and of T follows from the spectral mapping theorem (see [9, Theorem 4.2]). Observe that if estimate (41) holds for both T and $-T$ then (41) satisfies the condition of Theorem 4.8, so T and $-T$ generate the semigroups $\mathcal{U}^+(t)$ and $\mathcal{U}^-(t)$, respectively. Observe that the approximations

$$\mathcal{U}_{s_0}^+(t) := \sum_{n\geqslant 0} \frac{1}{n!} t^n \left(\mathcal{Y}_L^+(s_0)\right)^n, \qquad \mathcal{U}_{s_0}^-(t) := \sum_{n\geqslant 0} \frac{1}{n!} t^n \left(\mathcal{Y}_L^-(s_0)\right)^n$$

commute and as a consequence $\mathcal{U}^+(t)$ and $\mathcal{U}^-(t)$ commute. Thus

$$\mathcal{Z}(t) = \mathcal{U}^+(t)\mathcal{U}^-(t)$$

is a semigroup defined on $[0, \infty)$. Considering $v \in \mathcal{D}(T) = \mathcal{D}(-T)$ we have

$$\frac{1}{t}\left[\mathcal{Z}(t)v - v\right] = \mathcal{U}^-(t)\frac{\mathcal{U}^+(t)v - v}{t} + \frac{\mathcal{U}^-(t)v - v}{t}$$

and taking the limit for $t \to 0$ we get

$$\lim_{t\to 0} \frac{1}{t}\left[\mathcal{Z}(t)v - v\right] = Tv - Tv = 0,$$

that is

$$\frac{d}{dt}\mathcal{Z}(t)v = 0,$$

which implies $\mathcal{Z}(t)v = v$ for all $v \in \mathcal{D}(T)$. By assumption $\mathcal{D}(T)$ is dense in V so $\mathcal{U}^-(t) = (\mathcal{U}^+(t))^{-1}$. Therefore we define

$$\mathcal{U}(t) = \begin{cases} \mathcal{U}^+(t) & \text{if } t \geqslant 0, \\ \mathcal{U}^-(t) & \text{if } t < 0. \end{cases}$$

Then $\mathcal{U}(t)$ is a strongly continuous group whose infinitesimal generator is T and the estimate $\|\mathcal{U}(t)\| \leqslant Me^{t|\omega|}$ holds. $\square$

6. Remarks on the use of the functional calculus associated to right regular functions

The Laplace transforms for strongly continuous quaternionic semigroups can be introduced using the functional calculus associated to right regular functions (in short we will refer to it as "the right case" of the functional calculus).

Theorem 6.1. *Let $\mathcal{U}(t)$ be a strongly continuous quaternionic semigroups and set*

$$\omega_0 := \lim_{t\to+\infty} \frac{1}{t} \ln\left\|\mathcal{U}(t)\right\|.$$

Assume that $\mathcal{U}(t)$ is generated by a linear quaternionic operator T and take $s \in \mathbb{H}$ such that $\mathrm{Re}[s] > \omega_0$. *Then we have that $s \in \rho_S(T)$ and the right S-resolvent operator is given by*

$$S_R^{-1}(s,T)v = \int_0^\infty e^{-ts}\mathcal{U}(t)v\,dt, \quad v \in V. \tag{42}$$

Proof. Since there are analogies with the left S-resolvent operator, we simply point out the main differences. Assume that $\omega_0 < \delta < \mathrm{Re}[s]$. By Proposition 3.6 there exists a positive constant M_δ such that

$$\left\|\mathcal{U}(t)\right\| \leqslant M_\delta e^{\delta t}, \quad \text{for } t \geqslant 0, \tag{43}$$

so the integral (42) exists for $\mathrm{Re}[s] > \omega_0$ and defines a bounded operator. We now define

$$G(s)v := \int_0^\infty e^{-ts}\mathcal{U}(t)v\,dt, \quad v \in V, \tag{44}$$

and we consider

$$G(s)T_h v := \int_0^\infty e^{-ts}\mathcal{U}(t)T_h v\,dt, \quad v \in V.$$

With analogous computations we deduce that G satisfies the equation

$$sG(s)v - G(s)Tv = \mathcal{I}v, \quad \text{for all } v \in \mathcal{D}(T).$$

So G satisfies Eq. (16) and $G(s)v = S_R^{-1}(s,T)v$ for all v in $\mathcal{D}(T)$ which is dense in V. By Theorem 2.18 operator $S_R^{-1}(s,T)T$ in Eq. (16) admits a continuous extension to V, this means that $G(s)$ satisfy also (17):

$$sG(s)v - \widehat{G(s)T}v = \mathcal{I}v, \quad \text{for all } v \in V,$$

from which we get $G(s)v = S_R^{-1}(s,T)v$ for all $v \in V$. $\square$

Theorem 6.2 *(Hille–Yosida–Phillips: necessary condition for the right case). Let T be a linear quaternionic operator with dense domain and whose spectrum lies in the half space* $\mathrm{Re}[s] \leqslant \omega$, *where $\omega \in \mathbb{R}$. Let $\mathcal{U}(t)$ be a strongly continuous semigroup for $t \geqslant 0$ and suppose that there exist $M > 0$ and $\omega \in \mathbb{R}$ such that:*

$$\left\|\mathcal{U}(t)\right\| \leqslant Me^{\omega t}, \quad t \geqslant 0.$$

Suppose that

$$S_R^{-1}(s,T)v = \int_0^\infty e^{-st}\mathcal{U}(t)v\,dt, \quad v \in V.$$

Then we have the following estimate

$$\left\|(T-\bar{s}\mathcal{I})^{n*}Q_s(T)^n\right\| \leqslant \frac{M}{(\mathrm{Re}[s]-\omega)^n}, \quad n \in \mathbb{N}.$$

Proof. It follows with analogous computations of Theorem 4.3 but in this case we do not have to consider the extension (33) because the derivatives with respect to s of the right S-resolvent operator give a family of bounded operators on V directly. In fact, the operators $T^{n-k}Q_s(T)^n$ are bounded for $s \in \rho_S(T)$ and so also the operators

$$(T-\bar{s}\mathcal{I})^{n**}Q_s(T)^n = \sum_{n=0}^{n}\binom{n}{k}(\bar{s})^k T^{n-k}Q_s(T)^n,$$

where

$$(T-\bar{s}\mathcal{I})^{n**} = \sum_{n=0}^{n}\binom{n}{k}(\bar{s})^k T^{n-k}$$

are bounded on V. $\square$

Remark 6.3. In the sufficient condition in the Hille–Yosida–Phillips Theorem 4.8 we cannot distinguish between the right and the left case because $S_L^{-1}(s_0,T) = S_R^{-1}(s_0,T)$ for s_0 real number.

7. Some considerations on the case of left linear quaternionic operators

In this section, we will show in which sense the proofs of the theorems of the previous sections have to be modified when dealing with left linear, unbounded quaternionic operators. For more details we refer the reader to our paper [10].

Consider a bilateral vector space V. A map $T: V \to V$ is said to be a left linear quaternionic operator if

$$T(u+v) = T(u) + T(v), \qquad T(su) = sT(u), \quad \text{for all } s \in \mathbb{H}, \text{ and for all } u, v \in V.$$

The set $\mathrm{End}^L(V)$ of left linear operators on V is both a left and a right vector space on $\mathbb{H}$ with respect to the operations (see [2, Proposition 4.4])

$$(Ts)(v) := T(v)s, \qquad (sT)(v) := T(vs), \quad \text{for all } s \in \mathbb{H}, \text{ and for all } v \in V. \tag{45}$$

To explain the relation between $\mathrm{End}^L(V)$ and the set $\mathrm{End}^R(V)$ of right linear quaternionic operators, we recall the following:

Definition 7.1. Given a ring $(R, +, *)$ where $+, *$ denote the addition and the multiplication operations, respectively, the opposite ring $(R^{op}, +^{op}, *^{op})$ has the same underlying set, i.e. $R^{op} = R$ and the same additive structure while the multiplication $*^{op}$ is defined by $r *^{op} s := s * r$.

The following result can be found for example in [2, Section 4]:

Proposition 7.2. *The two rings* $\mathrm{End}^R(V)$ *and* $\mathrm{End}^L(V)$ *with respect to the addition and composition of operators are opposite rings of each other.*

Remark 7.3. We have that V is a module on the left on the ring $\mathrm{End}^R(V)$ and it is a module on the right on the ring $\mathrm{End}^L(V)$ (see [2, Section 4]). For this reason, the action of a right linear operator T on a vector $v \in V$ is often denoted in the literature by Tv while if T is a left linear operator, its action on v is denoted by vT. In light of this notation, the properties (45) can be written as

$$v(Ts) = (vT)s, \qquad v(sT) = (vs)T, \quad \text{for all } s \in \mathbb{H}, \text{ and for all } v \in V.$$

This notation for left linear operators is particularly useful when dealing with the composition of left linear operators.

In the sequel $\mathcal{K}^L(V)$ denotes the analogue of $\mathcal{K}(V)$ in Definition 2.13 for left linear quaternionic operators.

Definition 7.4 *(The S-resolvent operators for unbounded left linear operators).* Let V be a bilateral quaternionic Banach space, let $T \in \mathcal{K}^L(V)$ and $s \in \rho_S(T)$. We define the left S-resolvent operator as

$$vS_L^{-1}(s,T) := -vQ_s(T)(T - \mathcal{I}\bar{s}), \quad \text{for all } v \in V. \tag{46}$$

We define the right S-resolvent operator as

$$vS_R^{-1}(s,T) := -v(T - \mathcal{I}\bar{s})Q_s(T), \quad \text{for all } v \in \mathcal{D}(T), \tag{47}$$

and we will call

$$v\widehat{S}_R^{-1}(s,T) = vQ_s(T)\bar{s} - vQ_s(T)T, \quad \text{for all } v \in V, \tag{48}$$

the extended right S-resolvent operator.

Theorem 7.5 *(The S-resolvent equations). Let V be a bilateral quaternionic Banach space. If $T \in \mathcal{K}^L(V)$ and $s \in \rho_S(T)$, then the left S-resolvent operator satisfies the equations*

$$vS_L^{-1}(s,T)s - vTS_L^{-1}(s,T) = v\mathcal{I}, \quad \textit{for all } v \in \mathcal{D}(T), \tag{49}$$

$$v\widehat{S}_L^{-1}(s,T)s - v\widehat{TS_L^{-1}}(s,T) = v\mathcal{I}, \quad \textit{for all } v \in V. \tag{50}$$

Moreover, the right S-resolvent operator satisfies the equations

$$vsS_R^{-1}(s,T) - vS_R^{-1}(s,T)T = v\mathcal{I}, \quad \textit{for all } v \in \mathcal{D}(T), \tag{51}$$

$$vs\widehat{S}_R^{-1}(s,T) - v\big(\widehat{S}_R^{-1}(s,T)T\big) = v\mathcal{I}, \quad \textit{for all } v \in V. \tag{52}$$

To conclude, we summarize the main changes in our theory which are necessary when dealing with left linear operators.

(1) Consider Theorem 4.2. In (23), the extended left S-resolvent operator $\widehat{S}_L^{-1}(s,T)v$ defined in (12) (for all $v \in V$) has to be replaced by the left S-resolvent operator $vS_L^{-1}(s,T)$ defined in (46) (for all $v \in V$). Eq. (26), which in the proof turns out to be the S-resolvent equation (15), for left linear operators becomes (50).
(2) The replacements in Theorem 4.2 described above, have to be repeated for the right S-resolvent operators $\widehat{S}_R^{-1}(s,T)$ and $S_R^{-1}(s,T)$ discussed in Section 6.
(3) Consider Theorem 4.3. In condition (30) we have to replace $\widehat{S}_L^{-1}(s,T)$ by $S_L^{-1}(s,T)$ as described in point (1). The operator defined in (29) and appearing in estimate (31) has to be replaced by

$$\mathcal{Q}_s(T)^n(T - \bar{s}\mathcal{I})^{n*} = \mathcal{Q}_s(T)^n \sum_{n=0}^{n} \binom{n}{k} T^{n-k}(\bar{s})^k.$$

Finally, we point out that the need to use extensions of operators, like in the case of the extended S-resolvent operator $\widehat{S}_L^{-1}(s,T)v$ and T right linear, or the use of $vS_L^{-1}(s,T)$ when T is left linear, is due to the fact the action of those operators has to be defined for all $v \in V$.

References

[1] S. Adler, Quaternionic Quantum Field Theory, Oxford University Press, 1995.
[2] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, second ed., Springer-Verlag, New York, 1992.
[3] F. Brackx, R. Delanghe, F. Sommen, Clifford Analysis, Pitman Res. Notes Math., vol. 76, 1982.
[4] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, Extension results for slice regular functions of a quaternionic variable, Adv. Math. 222 (2009) 1793–1808.
[5] F. Colombo, G. Gentili, I. Sabadini, A Cauchy kernel for slice regular functions, Ann. Global Anal. Geom. 37 (2010) 361–378.
[6] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, Non commutative functional calculus: bounded operators, Complex Anal. Oper. Theory 4 (2010) 821–843.
[7] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, Non commutative functional calculus: unbounded operators, J. Geom. Phys. 60 (2010) 251–259.
[8] F. Colombo, I. Sabadini, A structure formula for slice monogenic functions and some of its consequences, in: Hypercomplex Analysis, in: Trends Math., Birkhäuser, 2009, pp. 101–114.

[9] F. Colombo, I. Sabadini, On some properties of the quaternionic functional calculus, J. Geom. Anal. 19 (2009) 601–627.
[10] F. Colombo, I. Sabadini, On the formulations of the quaternionic functional calculus, J. Geom. Phys. 60 (2010) 1490–1508.
[11] F. Colombo, I. Sabadini, The Cauchy formula with s-monogenic kernel and a functional calculus for noncommuting operators, J. Math. Anal. Appl. 373 (2011) 655–679.
[12] F. Colombo, I. Sabadini, F. Sommen, D.C. Struppa, Analysis of Dirac Systems and Computational Algebra, Prog. Math. Phys., vol. 39, Birkhäuser, Boston, 2004.
[13] F. Colombo, I. Sabadini, D.C. Struppa, A new functional calculus for noncommuting operators, J. Funct. Anal. 254 (2008) 2255–2274.
[14] F. Colombo, I. Sabadini, D.C. Struppa, Slice monogenic functions, Israel J. Math. 171 (2009) 385–403.
[15] F. Colombo, I. Sabadini, D.C. Struppa, An extension theorem for slice monogenic functions and some of its consequences, Israel J. Math. 177 (2010) 369–389.
[16] F. Colombo, I. Sabadini, D.C. Struppa, Duality theorems for slice hyperholomorphic functions, J. Reine Angew. Math. 645 (2010) 85–104.
[17] F. Colombo, I. Sabadini, D.C. Struppa, Noncommutative Functional Calculus, Progr. Math., vol. 289, Birkhäuser, Basel, 2011.
[18] N. Dunford, J. Schwartz, Linear Operators, Part I: General Theory, J. Wiley and Sons, 1988.
[19] G. Gentili, D.C. Struppa, A new theory of regular functions of a quaternionic variable, Adv. Math. 216 (2007) 279–301.
[20] R. Ghiloni, A. Perotti, Slice regular functions on real alternative algebras, Adv. Math. 226 (2011) 1662–1691.
[21] B. Jefferies, Spectral Properties of Noncommuting Operators, Lecture Notes in Math., vol. 1843, Springer-Verlag, Berlin, 2004.
[22] B. Jefferies, A. McIntosh, The Weyl calculus and Clifford analysis, Bull. Aust. Math. Soc. 57 (1998) 329–341.
[23] B. Jefferies, A. McIntosh, J. Picton-Warlow, The monogenic functional calculus, Studia Math. 136 (1999) 99–119.
[24] A. McIntosh, A. Pryde, A functional calculus for several commuting operators, Indiana Univ. Math. J. 36 (1987) 421–439.
[25] W. Rudin, Functional Analysis, McGraw–Hill Ser. Higher Math., McGraw–Hill Book Co., New York, Düsseldorf, Johannesburg, 1973.